Therefore

$$a^2 + b^2 + \left(\frac{a^2 + b^2 - c^2}{2c}\right)^2 = \left(\frac{a^2 + b^2 + c^2}{2c}\right)^2,$$

whatever be the values of a, b, c.

1. Taking $a = 1$, $b = 2$, $c = 1$, we find

$$1^2 + 2^2 + 2^2 = 3^2,$$

which is the *smallest* rational parallelepiped.

2. Taking $a = 2$, $b = 3$, $c = 1$, we have

$$2^2 + 3^2 + 6^2 = 7^2,$$

which is the smallest rational parallelepiped having its edges all different.

3. Taking $a = 1$, $b = 4$, $c = 1$, we get

$$1^2 + 4^2 + 8^2 = 9^2.$$

4. Let $a = 2$, $b = 6$, $c = 2$, and we have

$$2^2 + 6^2 + 9^2 = 11^2.$$

5. If $a = 3$, $b = 4$, $c = 1$, we get

$$3^2 + 4^2 + 12^2 = 13^2,$$

which, on p. 269, is stated to be "the smallest rational parallelopiped."

6. If we take $a = 8$, $b = 9$, $c = 5$, we will get

$$8^2 + 9^2 + 12^2 = 17^2.$$

7. Taking $a = 4$, $b = 8$, $c = 2$, we have

$$4^2 + 8^2 + 19^2 = 21^2.$$

8. If $a = 3$, $b = 4$, $c = 3$, then we get

$$12^2 + 15^2 + 16^2 = 25^2.$$

And so on, there being an infinite number of parallelopipeds whose edges and solid diagonals are rational integers.

The condition $x^2 + y^2 + z^2 = \square$ can be satisfied in many ways.—See *Mathematical Magazine,* Vol. II., No. 5 (October, 1891), pp. 71–74. Other methods are given in a forthcoming paper which will appear in the third part of No. 12, Vol. II., of the *Mathematical Magazine.*

416A. Proposed by H. O. HANSON, East Elmhurst, N. Y.

Find the nth term and the sum of n terms of the series obeying the relation $u_i = u_{i-1} + 2u_{i-2}$ in terms of n and the first two terms, u_1 and u_2, these two terms being arbitrary.

This problem was incorrectly numbered 416.

SOLUTION BY S. A. JOFFE, New York City.

Adding u_{i-1} to both members of the given relation, we obtain the following recurring formula:

$$u_i + u_{i-1} = 2(u_{i-1} + u_{i-2}). \tag{1}$$

If we replace i successively by n, $n-1$, $n-2$, $\cdots$, 3 and multiply the resulting $n-2$ equations, we have, after cancellation:

$$u_n + u_{n-1} = 2^{n-2}(u_2 + u_1). \tag{2}$$

Similarly

$$u_{n-2} + u_{n-3} = 2^{n-4}(u_2 + u_1),$$

$$u_{n-4} + u_{n-5} = 2^{n-6}(u_2 + u_1),$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot,$$

the last equation of this type being $u_2 + u_1 = u_2 + u_1$, or $u_3 + u_2 = 2(u_2 + u_1)$, according as n is even or odd.

Adding all these equations, we have:

for $n =$ even, $\sum_{i=1}^{n} u_i = (2^{n-2} + 2^{n-4} + 2^{n-6} + \cdots + 1)(u_2 + u_1)$,

or

$$\sum_{i=1}^{n} u_i = \tfrac{1}{3}(2^n - 1)(u_2 + u_1); \tag{3}$$

and for $n =$ odd,

$$\sum_{i=1}^{n} u_i = (2^{n-2} + 2^{n-4} + 2^{n-6} + \cdots + 2)(u_2 + u_1) + u_1 = \tfrac{2}{3}(2^{n-1} - 1)(u_2 + u_1) + u_1,$$

or

$$\sum_{i=1}^{n} u_i = \tfrac{1}{3}(2^n - 2)(u_2 + u_1) + u_1. \tag{4}$$

Formulæ (3) and (4) express the sum of the first n terms, as required, in terms of n and the first two terms; the nth term u_n is found by subtracting $\sum_{i=1}^{n-1} u_i$ from $\sum_{i=1}^{n} u_i$, the result being for $n =$ even,

$$u_n = \tfrac{1}{3}(2^n - 1)(u_2 + u_1) - [\tfrac{1}{3}(2^{n-1} - 2)(u_2 + u_1) + u_1]$$
$$= \tfrac{1}{3}(2^{n-1} + 1)(u_2 + u_1) - u_1;$$

and for $n =$ odd, $u_n = [\tfrac{1}{3}(2^n - 2)(u_2 + u_1) + u_1] - \tfrac{1}{3}(2^{n-1} - 1)(u_2 + u_1)$

$$= \tfrac{1}{3}(2^{n-1} - 1)(u_2 + u_1) + u_1.$$

The last two formulæ may be combined into one, for n in general,

$$u_n = \tfrac{1}{3}[2^{n-1} + (-1)^n](u_2 + u_1) + (-1)^{n-1}u_1. \tag{5}$$

Also solved by A. M. HARDING, A. H. HOLMES, HORACE OLSON, and the PROPOSER.